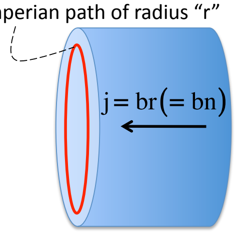
Problem 30.36

This is so fun (physics teachers across the country live for stuff like this, which is kind of sad and may explain why so few of them are married . . . and for those of you who couldn't tell, that was a joke . . .)!!! If you will remember back to your electric-field deriving days when Gauss's Law was the love of your life, it was possible to have a volume charge density inside a Gaussian surface that was not constant but, rather, varied with the distance out (i.e., a function that looked like $\rho = k_1 r$, where there is no charge at the center at r = 0 and the amount of *charge per unit volume* increased as you proceeded toward the edge). This is the magnetic fields version of that problem. It is attacked similarly.

As usual, we start by defining an Amperian path (that is shown on the sketch).

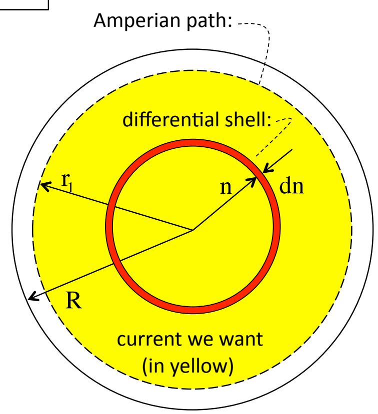
And the first thing to notice about *that* is the notational glitch. Ampere's Law usually uses the variable "r" for the radius of the Amperian path. But in this problem, the variable "r" is also used in conjunction with the current density function "j." To keep from getting confused, I'm going to let "n" be an arbitrary distance from the cylinder's center and write:



$$j = bn \text{ (amps per square meter)}$$

End view:

Because the *current per unit area* is increasing as we get farther from the center, we need to define a differentially thin circular strip of radius "n" and thickness "dn," determine the amount of current that flows through that strip, then integrate to determine the total amount of current that passes through the entire Amperian path. The sketch defines the quantities with which we will be working. As such, Ampere's Law yields:



(As this is a sizeable procedure, I'm putting it on the next page.)

End view:

$$\oint \vec{B} \bullet d\vec{l} = \mu_o i_{thru}$$

$$\Rightarrow \quad B\oint dl = \mu_o \int_{n=0}^{r_1} j\,dA$$

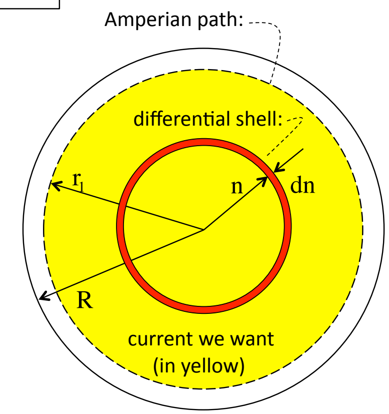
$$\Rightarrow \quad B(2\pi r_1) = \mu_o \int_{n=0}^{r_1} (bn)(2\pi n\ dn)$$

$$\Rightarrow \quad B(2\pi r_1) = 2\pi b\mu_o \int_{n=0}^{r_1} n^2\,dn$$

$$\Rightarrow \quad B(2\pi r_1) = 2\pi b\mu_o \left.\frac{n^3}{3}\right|_{n=0}^{r_1}$$

$$\Rightarrow \quad B(2\pi r_1) = 2\pi b\mu_o \frac{r_1^3}{3}$$

$$\Rightarrow \quad B = \frac{\mu_o b}{3} r_1^2$$



End view:

b.) for $r_2 > R$:

The only difference between this problem and the previous one is that the current integral terminates at n = R (i.e., where the current ends) and the Amperian path is outside the cylinder. With that, we can write:

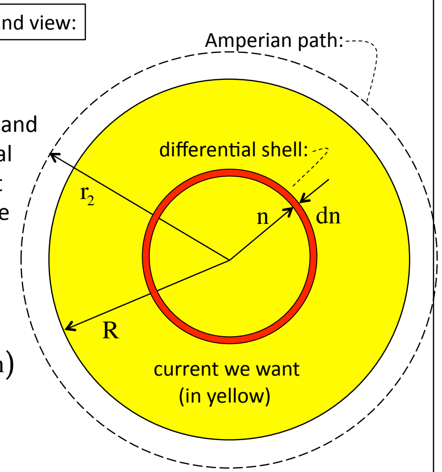
$$B\oint dl = \mu_o \int_{n=0}^{r_1} j\,dA$$

$$\Rightarrow \quad B(2\pi r_2) = \mu_o \int_{n=0}^{R} (bn)(2\pi n\ dn)$$

$$\Rightarrow \quad B(2\pi r_2) = 2\pi b\mu_o \left.\frac{n^3}{3}\right|_{n=0}^{R}$$

$$\Rightarrow \quad B(2\pi r_2) = 2\pi b\mu_o \frac{R^3}{3}$$

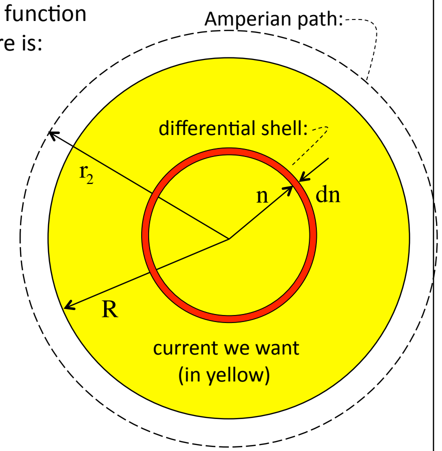
$$\Rightarrow \quad B = \frac{\mu_o b}{3 r_2} R^3$$



Additional comment: Notice that the field function (in its most general form) for inside the wire is:

$$B = \left(\frac{\mu_o b}{3}\right) r^2$$

According to this, when r = 0 the B-fld is zero. This is good as it is what might be expected of the system.



Similarly for outside the wire:

$$B = \left(\frac{\mu_o b R^3}{3}\right)\left(\frac{1}{r}\right)$$

This, again, is not surprising—we would expect the field to go to zero as r goes to infinity.

Lastly, at r = R, the two functions predict the same field. Also a good sign!

$$B_{insideFctAtR} = \left(\frac{\mu_o b}{3}\right) R^2 \quad \text{and} \quad B_{outsideFctAtR} = \left(\frac{\mu_o b R^3}{3}\right)\left(\frac{1}{R}\right) = \frac{\mu_o b R^2}{3}$$